ALGEBRA (INDIVIDUAL)

Problem 1.

(1) Classify the groups of order 8.
(2) For each finite group of order 8, classify the irreducible finite dimensional representations over $\mathbb{C}$.

Problem 2. For each integer $m > 1$, let $\mathcal{K}_m = \mathbb{Q}(e^{\frac{2\pi i}{m}})$.

(1) Prove that the polynomial $x^4 + x^3 + x^2 + x + 1$ is irreducible over $\mathbb{Q}$.
(2) Prove that $\mathcal{K}_5$ is a Galois extension of $\mathbb{Q}$ whose Galois group is cyclic of order 4.
(3) Prove that $\mathcal{K}_5 \supseteq \mathbb{Q}(\cos \frac{2\pi}{5})$.
(4) Prove that $\mathcal{K}_{20}$ is a Galois extension of $\mathbb{Q}$ whose Galois group is isomorphic to $(\mathbb{Z}/20\mathbb{Z})^\times$.

(Recall that $(\mathbb{Z}/20\mathbb{Z})^\times$ is the group of units of the ring $\mathbb{Z}/20\mathbb{Z}$.)

Problem 3. Let f be a nonconstant polynomial in $\mathbb{C}[x]$ and $R = \mathbb{C}[x]/(f)$. Show that the following are equivalent:

(1) R has no nonzero nilpotent element.
(2) any finitely generated indecomposable R-module is projective.